1962

Essential fixed points

Donald Lee Schmidt
Iowa State University

This dissertation has been microfilmed exactly as received 63-1595

SCHMIDT, Donald Lee, 1930-
ESSENTIAL FIXED POINTS.

Iowa State University of Science and Technology
Ph.D., 1962
Mathematics

University Microfilms, Inc., Ann Arbor, Michigan

ESSENTIAL FIXED POINTS

by

Donald Lee Schmidt

A Dissertation Submitted to the
Graduate Faculty in Partial Fulfillment of
The Requirements for the Degree of
DOCTOR OF PHILOSOPHY

Major Subject: Mathematics

Approved:

Signature was redacted for privacy.

In Charge of Major Work

Signature was redacted for privacy.

Head of Major Department

Signature was redacted for privacy.

Dean of Graduate College

Iowa State University
Of Science and Technology
Ames, Iowa

1962

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I. INTRODUCTION

A space X has the fixed point property if and only if every continuous mapping on X into X has at least one fixed point.

Let (X,d) be a compact metric space with the fixed point property. Denote by X^X the set of all continuous mappings on X into X metrized by setting $\rho(f,g) = \sup\{d(f(x),g(x)) | x \in X\}$. (X^X,ρ) is a complete metric space.

Let p be a fixed point of $f \in X^X$. p is an essential fixed point of f if and only if corresponding to each neighborhood U of p there is an $\epsilon > 0$ such that if $g \in X^X$ and $\rho(f,g) < \epsilon$ then g has a fixed point in U.

The concept of essentiality for a fixed point is a stability property which is analogous to the notion of a stable value for a function. See (3). The above definition is given by M. K. Fort, Jr. (1) in a paper published in the American Journal of Mathematics in 1950. Subsequently two other papers dealing with generalizations of this concept appeared in the literature, one in 1952 by Kinoshita (5) and the other by O'Neill (7).

A problem in the theory of essential fixed points is to find conditions under which a mapping will have essential fixed points. Fort proves two existence theorems. The first states that if a mapping has a unique fixed point, then that point is essential. By further restricting the space to be a topological n-manifold he shows that if the set of fixed points is totally disconnected then at least one fixed point must be essential.

When a mapping has no essential fixed points it may have a component of its set of fixed points which has the property of being essential. A component C of the set of fixed points of f is essential if and only if corresponding to each neighborhood U of C there is a neighborhood N of f such that if $g \in N$ then g has a fixed point in U. Kinoshita proves that if X is an absolute retract (a homeomorphic image of a retract of the Hilbert cube and hence metric) then every continuous mapping on X into X has an essential component of its set of fixed points.

O'Neill defines an essential set and considers mainly the problem of locating those sets which are essential with respect to a given mapping. If an essential set is a component of the set of fixed points then it is essential in the former sense.

It is the purpose of this paper to show that results concerning essential fixed points and components can be obtained in the more general setting of a compact Hausdorff space. Use is made of the fact that the topology of a compact Hausdorff space is a uniform topology. It turns out that such spaces have enough structure to enable one to obtain, in some cases, results just as strong as for metric spaces.

Chapter III of this paper contains theorems giving conditions under which a mapping will have only essential fixed points. This study is motivated by Fort's result that for a metric space such mappings form an everywhere dense set in X^X. Also in this section an example is given of a non-metric compact Hausdorff space with the fixed point property. Two theorems relating to essential fixed points are proved for this space. In Chapter IV the result of Kinoshita is generalized to include a certain class of non-metric absolute retracts. The last section deals with the behavior of the mapping in the neighborhood of an essential component of the set of fixed points and the problem of recognizing essential fixed points and components.

II. PRELIMINARIES

Throughout this paper use is made of the theory of uniform spaces. This chapter contains preliminary theorems and definitions which will be referred to in the following chapters. With the exception of Definition 2.13, Theorems 2.14 and 2.15, this material can be found in chapters six and seven of Kelley, General Topology (4). Standard notation is used throughout and in most cases agrees with that used by Kelley. As in Kelley, a neighborhood of x will mean a set containing an open set containing x.

Definition 2.1: (i) Let X be a set. A relation U in X is a subset of $X \times X$, or a set of ordered pairs (x,y).

(ii) If U is a relation then the inverse of U is defined by the equation $U^{-1} = \{(x,y) \mid (y,x) \in U\}$.

(iii) If $U = U^{-1}$ then U is symmetric.

(iv) If U and V are relations then their composition $U \cdot V$ is defined by the equation $U \cdot V = \{(x,y) \mid$ for some z, $(x,z) \in V$ and $(z,y) \in U\}$.

(v) The identity relation, or the diagonal, is defined by the equation $\Delta(X) = \{(x,x) \mid x \in X\}$.

(vi) If $A \subset X$ and U is a relation, then $U[A] = \{y \mid (x,y) \in U$ for some $x \in A\}$. If $A = \{x\}$, then $U[\{x\}] = U[x] = \{y \mid (x,y) \in U\}$.

<u>Definition 2.2</u>: A uniformity for a set X is a non-empty family $\mathcal{U}$ of subsets of $X \times X$, or relations in X, such that

(i) each member of $\mathcal{U}$ contains the diagonal;

(ii) if $U \in \mathcal{U}$, then $U^{-1} \in \mathcal{U}$;

(iii) if $U \in \mathcal{U}$, then $V \cdot V \subset U$ for some $V \in \mathcal{U}$;

(iv) if U and V are members of $\mathcal{U}$, then $U \cap V \in \mathcal{U}$;

(v) if $U \in \mathcal{U}$ and $U \subset V \subset X \times X$, then $V \in \mathcal{U}$. The pair $(X, \mathcal{U})$ is a uniform space.

<u>Definition 2.3</u>: A subfamily B of a uniformity $\mathcal{U}$ is a base for $\mathcal{U}$ if and only if each member of $\mathcal{U}$ contains a member of B.

<u>Theorem 2.1</u>: A family B of subsets of $X \times X$ is a base for some uniformity for X if and only if

(i) each member of B contains the diagonal;

(ii) if $U \in B$, then U^{-1} contains a member of B;

(iii) if $U \in B$, then $V \cdot V \subset U$ for some $V \in B$; and

(iv) the intersection of two members of B contains a member.

<u>Definition 2.4</u>: If $(X, \mathcal{U})$ is a uniform space the topology $\mathcal{T}$ of the uniformity $\mathcal{U}$, or the uniform topology, is the family of all subsets T of X such that for each $x \in T$ there is a $U \in \mathcal{U}$ such that $U[x] \subset T$.

<u>Theorem 2.2</u>: If B is a base for the uniformity $\mathcal{U}$ then for each x the family of sets $U[x]$ for $U \in B$ is a base for the

neighborhood system of x.

Theorem 2.3: If U is a member of the uniformity $\mathcal{U}$, then the interior (in the product uniform topology) of U is also a member; consequently the family of all open symmetric members of $\mathcal{U}$ is a base for $\mathcal{U}$.

Theorem 2.4: If $U \in \mathcal{U}$ is open in $X \times X$, then for each $x \in X$, U[x] is open in X.

Theorem 2.5: If $(X, \mathcal{T})$ is a compact regular topological space then the family of all neighborhoods of the diagonal is a uniformity for X and $\mathcal{T}$ is the uniform topology.

Definition 2.5: If f is a function on a uniform space $(X, \mathcal{U})$ to a uniform space $(Y, \mathcal{V})$, then f is uniformly continuous if and only if for each $V \in \mathcal{V}$ there is a $U \in \mathcal{U}$ such that $(f(x), f(y)) \in V$ whenever $(x,y) \in U$.

Definition 2.6: A net $\{S_n, n \in D\}$ in the uniform space $(X, \mathcal{U})$ is a Cauchy net if and only if for each member $U \in$ there is a member $N \in D$ such that $(S_n, S_m) \in U$ whenever n and m follow N in the ordering of D.

Definition 2.7: A uniform space is complete if and only if every Cauchy net in the space converges to a point in the space.

Definition 2.8: A uniform space $(X, \mathcal{U})$ is totally bounded if and only if for each $U \in \mathcal{U}$ there is a finite subset F of X such that $U[F] = X$.

<u>Definition 2.9</u>: A net $\{T_m, m \in E\}$ is a subnet of a net $\{S_n, n \in D\}$ if and only if there is a function N on E with values in D such that

(i) $T = S \cdot N$, or equivalently, $T_i = S_{N_i}$, for each $i \in E$; and

(ii) for each $n \in D$ there is $m \in E$ with the property that, if $p \geq m$, then $N_p \geq n$.

<u>Theorem 2.6</u>: A uniform space $(X, \mathcal{U})$ is totally bounded if and only if each net in X has a Cauchy subnet.

<u>Theorem 2.7</u>: A uniform space is compact if and only if it is totally bounded and complete.

<u>Theorem 2.8</u>: Let A be a compact subset of a uniform space $(X, \mathcal{U})$. Then each neighborhood of A contains a neighborhood of the form $U[A]$ for some $U \in \mathcal{U}$.

Let Y be a subset of a uniform space $(X, \mathcal{U})$. For each $U \in \mathcal{U}$, let $U' = U \cap Y \times Y$. The family of subsets $\{U \cap Y \times Y | U \in \mathcal{U}\}$ is a uniformity for Y.

<u>Definition 2.10</u>: The uniformity $\mathcal{V} = \{U \cap Y \times Y | U \in \mathcal{U}\}$ is called the relativization of $\mathcal{U}$ to Y, or the relative uniformity for Y, and $(Y, \mathcal{V})$ is called a uniform subspace of

the space $(X, \mathcal{U})$.

Theorem 2.9: The topology of the relative uniformity $\mathcal{V}$ is the relative topology of $\mathcal{U}$.

Theorem 2.10: A uniform space is completely regular.

Theorem 2.11: A uniform space is metrizable if and only if it is Hausdorff and its uniformity has a countable base.

Definition 2.11: Let F be a family of functions on a set X to a uniform space $(Y, \mathcal{V})$. For each $V \in \mathcal{V}$ let

$$W(V) = \{(f,g) \mid (f(x), g(x)) \in V \text{ for each } x \in X\}.$$

Theorem 2.12: The family of all sets $W(V)$ for $V \in \mathcal{V}$ is a base for a uniformity $\mathcal{U}$ for F.

Definition 2.12: The family $\mathcal{U} = \{W(V) \mid V \in \mathcal{V}\}$ is called the uniformity of uniform convergence. The topology of $\mathcal{U}$ is called the topology of uniform convergence.

A base for the neighborhood system of $f \in F$ is the family of sets $W(V)[f] = \{g \mid (g(x), f(x)) \in V \text{ for each } x \in X\}$.

Theorem 2.13: Let F be the family of all continuous functions on a space X to a uniform space $(Y, \mathcal{V})$, and let $\mathcal{U}$ be the uniformity of uniform convergence. Then the family F is closed in the space of all functions on X to Y, and consequently $(F, \mathcal{U})$ is complete if $(Y, \mathcal{V})$ is complete.

Let $C(X)$ denote the set of all compact subsets of a uniform space $(X, \mathcal{U})$. A topology for $C(X)$ will be constructed which is analogous to the topology of the Hausdorff metric for metric spaces.

<u>Definition 2.13</u>: For each $U \in \mathcal{U}$, let
$M(U) = \{(K,K') \in C(X) \times C(X) | K' \subset U[K] \text{ and } K \subset U[K']\}$.

<u>Lemma 2.1</u>: If U and V are in $\mathcal{U}$ and $U \subset V$, then $M(U) \subset M(V)$.

<u>Proof</u>: Let $(K, K') \in M(U)$. Then $K \subset U[K']$ and $K' \subset U[K]$. Since $U \subset V$ it follows that $K' \subset V[K]$ and $K \subset V[K]$. Hence $(K, K') \in M(V)$.

<u>Theorem 2.14</u>: The family $\{M(U) | U \in \mathcal{U}\}$ is a base for a uniformity $\mathcal{V}$ for $C(X)$.

<u>Proof</u>: The theorem will be proved by showing that the family $\{M(U)\}$ satisfies the conditions of Theorem 2.1. Let $B(C(X))$ denote the family $\{M(U)\}$.

(i) For each $U \in \mathcal{U}$ and $K \in C(X)$, $(K, K) \in M(U)$ and hence each $M(U)$ contains the diagonal.

(ii) From the definition of $M(U)$ it follows that $(M(U))^{-1} = M(U)$ for each $U \in \mathcal{U}$. Hence $(M(U))^{-1} \in B(C(X))$. In fact each $M(U)$ is symmetric.

(iii) For each $M(U) \in B(C(X))$ it must be shown that there exists a $M(V)$ such that $M(V) \cdot M(V) \subset M(U)$. Choose U and V in $\mathcal{U}$ such that $V \cdot V \subset U$. Then

$M(V) \cdot M(V) = \{(K, K'') \mid \text{for some } K', (K, K') \in M(V) \text{ and }$
$(K', K'') \in M(V)\}.$

$(K, K') \in M(V)$ implies that $K \subset V[K']$ and $K' \subset V[K]$. Similarly $(K', K'') \in M(V)$ implies that $K' \subset V[K'']$ and $K'' \subset V[K']$. Consequently $K \subset V \cdot V[K'']$ and $K'' \subset V \cdot V[K]$. Therefore

$$M(V) \cdot M(V) = \{(K, K'') \mid K \subset V \cdot V[K''] \text{ and } K'' \subset V \cdot V[K]\} = M(V \cdot V).$$

Now $V \subset V \cdot V \subset X \times X$ and by Definition 2.2 (v), $V \cdot V \in \mathcal{U}$. By Lemma 2.1, $M(V \cdot V) \subset M(U)$ since $V \cdot V \subset U$. Hence $M(V) \cdot M(V) \subset M(U)$.

(iv) Let $U, V \in \mathcal{U}$. Then

$$\begin{aligned} M(U) \cap M(V) &= \{(K,K') \mid K \subset U[K'], K' \subset U[K] \text{ and } K \subset V[K'], K' \subset V[K]\}. \\ &\supset \{(K, K') \mid K \subset (U \cap V)[K'] \text{ and } K' \subset (U \cap V)[K]\}. \\ &= M(U \cap V) \in B(C(X)). \end{aligned}$$

This completes the proof of the theorem.

The base $B(C(X))$ for $\mathcal{V}$ is equivalent to a base for a uniformity 2^U constructed by Michael (6). In fact if only symmetric members of $\mathcal{U}$ are used in the construction of these bases then they are identical. Michael proves the following theorem for $C(X)$.

<u>Theorem 2.15</u>: $C(X)$ is compact Hausdorff if and only if X is compact Hausdorff.

III. ESSENTIAL FIXED POINTS

Throughout this chapter, unless otherwise stated, X will be a compact Hausdorff space with the fixed point property. Since a compact Hausdorff space is regular it follows from Theorem 2.5 that X is a uniform space with a uniformity $\mathcal{U}$ consisting of all the neighborhoods of the diagonal. Let $B(X)$ denote the set of all open symmetric neighborhoods of the diagonal. By Theorem 2.3 $B(X)$ is a base for $\mathcal{U}$ and from Theorems 2.2 and 2.4 it follows that a base for the topology for X is the set of all open sets $U[x]$, for $U \in B(X)$ and $x \in X$.

Let X^X denote the set of all continuous mappings on X into X topologized by the topology of uniform convergence. See Definitions 2.11, 2.12 and Theorem 2.12.

Definition 3.1: Let p be a fixed point of $f \in X^X$. p is an essential fixed point of f if and only if corresponding to each neighborhood U of p there is a neighborhood N of f such that if $g \in N$, then g has a fixed point in U.

Let $C(X)$ denote the set of all compact subsets of X. Let $C(X)$ be given the topology of the uniformity defined in Definition 2.13. According to Theorem 2.15, with this topology, $C(X)$ is a compact Hausdorff space.

Definition 3.2: Let $f \in X^X$. Define

$F(f) = \{x \in X | f(x) = x\}$.

Theorem 3.1: For each $f \in X^X$, $F(f)$ is a compact subset of X.

This theorem is easily proved by showing that $F(f)$ is closed in X. The compactness follows from the fact that a closed subset of a compact set is compact.

Hence for each $f \in X^X$, $F(f) \in C(X)$ and F is a function on X^X into $C(X)$.

Definition 3.3: (i) F is upper semi-continuous (USC) at $f \in X^X$ if and only if corresponding to each $U \in \mathcal{U}$ there is a neighborhood N of f such that if $g \in N$ then $F(g) \subset U[F(f)]$.

(ii) F is lower semi-continuous (LSC) at $f \in X^X$ if and only if corresponding to each $U \in \mathcal{U}$ there is a neighborhood N of f such that if $g \in N$ then $F(f) \subset U[F(g)]$.

(iii) F is continuous at $f \in X^X$ if and only if corresponding to each neighborhood V of $F(f) \in C(X)$ there is a neighborhood N of f such that if $g \in N$ then $F(g) \in V$.

The straightforward proof of the following theorem will be omitted.

Theorem 3.2: F is continuous at $f \in X^X$ if and only if F is USC and LSC at f.

Lemma 3.1: Let $(X, \mathcal{U})$ be a Hausdorff uniform space and K a compact subset of X. Let f be a continuous mapping on X into

X such that $f(x) \neq x$ for every $x \in K$. Then there is a $W \in \mathcal{U}$ such that $f(x) \notin W[x]$ for every $x \in K$.

<u>Proof</u>: For every $x \in K$ there exist neighborhoods $N(x)$ and $M(f(x))$ such that $N(x) \cap M(f(x)) = \emptyset$. Since f is continuous, for each $x \in K$ there is a $U_x \in \mathcal{U}$ such that $U_x[x] \subset N(x)$ and $f(U_x[x]) \subset M(f(x))$. For each U_x choose a $V_x \in \mathcal{U}$ such that $V_x \cdot V_x^{-} \subset U_x$. The collection $\{V_x[x]\}$ is an open cover for K and since K is compact there is a finite set $x_1, x_2, \ldots, x_n$ in K such that

$$K \subset \bigcup_{i=1}^{n} V_{x_i} [x_i].$$

Choose $W \in \mathcal{U}$ such that

$$W \subset \bigcap_{i=1}^{n} V_{x_i}.$$

Let $y \in K$. Then $y \in V_{x_k} [x_k]$ for some k, $1 \leq k \leq n$. Hence

$W[y] \subset U_{x_k} [x_k] \subset N(x_k)$ and $f(y) \in f(U_{x_k} [x_k]) \subset M(f(x_k))$.

Since $N(x_k) \cap M(f(x_k)) = \emptyset$, $f(y) \notin W[y]$.

<u>Theorem 3.3</u>: F is USC on X^X to $C(X)$.

<u>Proof</u>: Let $f \in X^X$. Let U be an open set in X containing $F(f)$. X - U is a compact subset of a Hausdorff uniform space and hence by Lemma 3.1 there is a $V \in \mathcal{U}$ such that $f(x) \notin V[x]$ for each $x \in X - U$. Choose $V' \in B(X)$ such that $V' \subset V$. Then $f(x) \notin V'[x]$ and since V' is symmetric it is true that

$x \notin V'[f(x)]$. Choose $g \in X^X$ such that $g \in W(V')[f]$. For each $x \in X - U$, $g(x) \in V'[f(x)]$ but $x \notin V'[f(x)]$ and consequently $g(x) \neq x$. Therefore g has no fixed point in $X - U$ and it follows that $F(g) \subset U$.

<u>Theorem 3.4</u>: Each fixed point of $f \in X^X$ is essential if and only if F is LSC at f.

<u>Proof</u>: Suppose that each fixed point of f is essential. If $U \in \mathcal{U}$, choose $V \in B(X)$ such that $V \cdot V \subset U$. For each $p \in F(f)$ there is a neighborhood N_p of f such that if $g \in N_p$ then g has a fixed point in $V[p]$. Since $F(f)$ is compact there is a finite set $p_1, p_2, \ldots, p_n$ in $F(f)$ such that $F(f) \subset \bigcup_{i=1}^{n} V[p_i]$. Let $W = \bigcap_{i=1}^{n} N_{p_i}$. Let $g \in W$. Then g has a fixed point in each $V[p_i]$, $i = 1, 2, \ldots, n$. For each $p \in F(f)$, $p \in V[p_k]$ for some k, $1 \leq k \leq n$, and there is a $q \in F(g)$ such that $q \in V[p_k]$. By symmetry $p_k \in V[q]$ and since $V \cdot V \subset U$ it is true that $p \in U[q]$. Therefore every $p \in F(f)$ is in $U[q]$ for some $q \in F(g)$ and so $F(f) \subset U[F(g)]$. Thus F is LSC at f.

Suppose now that F is LSC at f. Let W be a neighborhood of $p \in F(f)$. Choose $U \in B(X)$ such that $U[p] \subset W$. Since F is LSC there exists a neighborhood N of f such that if $g \in N$ then $F(f) \subset U[F(g)]$. Hence $p \in U[q]$ for some $q \in F(g)$. By symmetry $q \in U[p] \subset W$ and so p is essential.

<u>Corollary 3.1</u>: Each fixed point of $f \in X^X$ is essential if and only if F is continuous at f.

<u>Proof</u>: If F is continuous at f then F is LSC by Theorem 3.2 and by Theorem 3.4 each fixed point of f is essential. If each fixed point of f is essential then by Theorem 3.4 F is LSC at f. The continuity of F follows from Theorems 3.3 and 3.2.

<u>Definition 3.4</u>: Let $U \in B(X)$. Let $A(U)$ be the set of all $f \in X^X$ such that for every neighborhood N of f and any $V \in B(X)$ there is a $g \in N$ such that $V[F(f)] \not\subset U[F(g)]$.

If F is not continuous at $f \in X^X$ then F is not LSC at F in view of Theorem 3.2 and hence f is in $A(U)$ for some $U \in B(X)$. The theorem and corollary following Lemma 3.2 show that F is not too badly discontinuous in the sense that if $U \in B(X)$ the criterion of continuity relative to U is satisfied almost everywhere in X^X.

<u>Lemma 3.2</u>: Let $f \in X^X$ and $U, V \in B(X)$ such that $V \cdot V \subset U$. Then there is a neighborhood N of f such that if $g \in N$ then $V[F(g)] \subset U[F(f)]$.

<u>Proof</u>: Since $\bar{F}$ is USC at f there is a neighborhood N of f such that if $g \in N$ then $F(g) \subset V[F(f)]$. Let $p \in V[F(g)]$. Then $p \in V[q]$ for some $q \in F(g)$ and $q \in V[r]$ for some $r \in F(f)$. Since $V \cdot V \subset U$ it follows that $p \in V \cdot V[r] \subset U[r]$. Therefore $V[F(g)] \subset U[F(f)]$.

Theorem 3.5: For every $U \epsilon B(X)$ the set $A(U)$ is nowhere dense in X^X.

Proof: The theorem will be proved by showing that $A(U)$ is a closed set in X^X which contains no non-empty open set.

Let f be a limit point of $A(U)$ and V be any member of $B(X)$. Choose $W \epsilon B(X)$ such that $W \cdot W \subset V$. By Lemma 3.2 there is an open neighborhood N of f such that if $g \epsilon N$ then $W[F(g)] \subset V[F(f)]$. Let $g \epsilon N \cap A(U)$ and choose $h \epsilon N$ such that $W[F(g)] \not\subset U[F(h)]$. This is possible since $g \epsilon A(U)$ and N is a neighborhood of g. Suppose now that $V[F(f)] \subset U[F(h)]$. Since $W[F(g)] \subset V[F(f)]$ this would imply that $W[F(g)] \subset U[F(h)]$ which contradicts the choice of h. Thus $f \epsilon A(U)$ and $A(U)$ is closed.

Assume that $A(U)$ contains a non-empty open set G. Choose a sequence $\{V_i\}$ in $B(X)$ such that $V_{i+1} \cdot V_{i+1} \subset V_i$, $i = 1, 2, \ldots$, and $V_1 \cdot V_1 \subset U$. Choose a point in G and call it f_1. By Lemma 3.2 and Definition 3.4 there exists a point $f_2 \epsilon G$ such that $V_2[F(f_2)] \subset V_1[F(f_1)]$ and $V_1[F(f_1)] \not\subset U[F(f_2)]$. In general if f_i has been chosen, select f_{i+1} such that $V_{i+1}[F(f_{i+1})] \subset V_i[F(f_i)]$ and $V_i[F(f_i)] \not\subset U[F(f_{i+1})]$. Suppose now that $V_j[F(f_j)] \subset U[F(f_i)]$ for $j < i$. Since $V_{i-1}[F(f_{i-1})] \subset \cdots \subset V_{j+1}[F(f_{j+1})] \subset V_j[F(f_j)]$ this would imply that $V_{i-1}[F(f_{i-1})] \subset U[F(f_i)]$ which contradicts the choice of the f_i. It is also true that $V_j[F(f_j)] \subset V_1[F(f_j)]$ and hence $V_1[F(f_j)] \not\subset U[F(f_i)]$ for $j < i$. It now follows that $F(f_j) \not\subset V_1[F(f_j)]$ for $j < i$. For suppose that for some $j < i$, $F(f_j) \subset \check{V}_1[F(f_i)]$. Since $V_1 \cdot V_1 \subset U$ this would imply that $V_1[F(f_j)] \subset \check{U}[F(f_i)]$ which contradicts the fact that $V_1[F(f_j)] \not\subset U[F(f_i)]$ for $j < i$. Assume now that the sequence, or net, $\{F(f_i), I\}$ (where I denotes the positive integers) has a Cauchy subnet $\{T_q, E\}$. Then since $V_1 \in B(X)$ there is a $p \in E$ such that $(T_m, T_n) \in M(V_1)$ whenever $m, n \geq p$. Choose $n > p$. By Definition 2.9 $T_n = F(f_{N_n})$ where $N_n \in I$ and there is a $q \in E$ such that $s \geq q$ implies that $N_s > N_n$. But given $q, n \in E$ there exists $m \in E$ such that $m \geq q$ and $m \geq n > p$. Hence $N_m > N_n$ and $(F(f_{N_m}), F(f_{N_n}) = (T_m, T_n) \subset M(\check{V}_1)$. This implies that $F(f_{N_n}) \subset V_1[F(f_{N_m})]$ and contradicts the fact that for $j < i$, $F(f_j) \not\subset \check{V}_1[F(f_i)]$. Consequently the sequence $\{F(f_i)\}$ has no Cauchy subnet which in view of Theorem 2.7 contradicts the total boundedness of C(X). Therefore A(U) contains no interior points and the theorem is proved.

<u>Corollary 3.2</u>: Let $U \in B(X)$. The set of all $f \in X^X$ for which there is a neighborhood N such that $g \in N$ implies that $F(g) \subset U[F(f)]$ and $F(f) \subset U[F(g)]$ is everywhere dense in X^X.

<u>Proof</u>: Let $f \in X^X - A(U)$. Since F is USC there is a neighborhood N of f such that $F(g) \subset U[F(f)]$ if $g \in N$. Suppose that for every neighborhood N' of f there is a $g' \in N'$ such that $F(f) \not\subset U[F(g)]$. Then for any $V \in B(X)$, $V[F(f)] \not\subset U[F(g)]$ which implies that $f \in A(U)$ and contradicts

the fact that $f \in X^X - A(U)$. Hence there is a neighborhood M of f such that if $g \in M$ then $F(f) \subset U[F(f)]$ and $F(f) \subset U[F(g)]$. Since $X^X - A(U)$ is everywhere dense the corollary follows.

If X is metric Fort's (1) result that the set of mappings all of whose fixed points are essential is dense in X^X follows from the fact that X is a uniform space whose uniformity has a countable base $\{U_i\}$, $i = 1, 2, 3, \cdots$. The points of discontinuity of F are then contained in

$\bigcup_{i=1}^{\infty} A(U_i)$. Since X^X is a complete metric space the

complement of this countable union of nowhere dense sets, consisting of the points of continuity of F, is dense in X^X. The result follows from Corollary 3.1. In view of Theorem 2.11 a generalization using this method does not seem possible if X is compact Hausdorff.

However the next theorem and corollary show that for a given $U \in B(X)$, mappings which have essential fixed points or all of whose fixed points are essential can, under certain conditions, be found in $X^X - A(U)$.

<u>Theorem 3.6</u>: Let $U \in B(X)$ and $f \in X^X - A(U)$. If p is a fixed point of f and no other fixed point of f is in $\overline{U[p]}$, then p is essential.

<u>Proof</u>: Since $f \in X^X - A(U)$ there is a neighborhood N of f such that for $g \in N$ it is true that $F(g) \subset U[F(f)]$ and

$F(f) \subset U[F(g)]$. By hypothesis $\overline{U[p]} \cap [F(f) - \{p\}] = \emptyset$ and since these sets are closed there is a $V' \in B(X)$ such that $V'[F(f) - \{p\}] \cap U[p] = \emptyset$. Let W be any neighborhood of p. Choose $U' \in B(X)$ such that $U'[p] \subset W$ and $U' \subset V'$. Since F is USC there is a neighborhood N' of f such that $g \in N'$ implies that $F(g) \subset U'[F(f)]$. Let $g \in N \cap N'$. Then $F(g) \subset U'[F(f)]$ and $F(f) \subset U[F(g)]$. Hence there is a point $q \in F(g)$ such that $p \in U[q]$ and by symmetry $q \in U[p]$. Since $F(g) \subset U'[F(f)]$ q must be in $U'[x]$ for some $x \in F(f)$. But $U'[x] \cap U[p] = \emptyset$ for $x \in F(f) - \{p\}$ since $U'[x] \subset V'[x]$. Therefore $x = p$ and $q \in U'[p] \subset W$. It follows that p is essential.

<u>Corollary 3.3</u>: If $f \in X^X - A(U)$ and if for each $p \in F(f)$ $\overline{U[p]}$ contains no other fixed points of f then the fixed points of f are all essential.

As a consequence of the compactness of $F(f)$ a mapping satisfying the condition of Corollary 3.3 can have only a finite number of fixed points.

In some cases fixed points of g "close" to essential fixed points of f inherit the property of being essential.

<u>Theorem 3.7</u>: Let $U \in B(X)$ and $f \in X^X$. If p is an essential fixed point of f then there is a neighborhood N of f such that if $g \in N$ has only one fixed point $q \in U[p]$, then q is essential.

<u>Proof</u>: Choose a neighborhood W of p such that $\overline{W} \subset U[p]$. Since p is essential there is an open neighborhood N of f such that if $g \in N$ then g has a fixed point q in W. Suppose that q is the only fixed point of g in $U[p]$. Then $F(g) - \{q\}$ and $\overline{W}$ are disjoint closed subsets of X and there is a $V \in B(X)$ such that $V[F(g) - \{q\}] \cap W = \emptyset$. Let W' be any neighborhood of q. Choose $V' \in B(X)$ such that $V' \subset V$ and $V'[q] \subset W'$. Let N' be a neighborhood of g such that $h' \in N'$ implies that $F(h') \subset V'[F(g)]$. Let $h \in N \cap N'$. It follows that $F(h) \subset V'[F(g)]$ and that h has a fixed point r in W. Now r must be in $V'[x]$ for some $x \in F(g)$. Suppose $x \in F(g) - \{q\}$. Then $r \in V'[F(g) - \{q\}]$. But since $V'[F(g) - \{q\}] \cap W = \emptyset$ this contradicts the fact that $r \in W$. Hence $r \in V'[q] \subset W'$ and so q is essential.

<u>Theorem 3.8</u>: Let $p_1, p_2, \ldots, p_n$ be essential fixed points of $f \in X^X$. There exists a neighborhood N of f such that if $g \in N$ has exactly n fixed points then they are all essential.

<u>Proof</u>: Choose $U \in B(X)$ such that $U[p_i] \cap U[p_j] = \emptyset$ for $i \neq j$, $i, j = 1, 2, \ldots, n$. Since the p_i are all essential it is possible to find a neighborhood N of f such that if $g \in N$ then g has a fixed point in each $U[p_i]$. Suppose $g \in N$ has exactly n fixed points. Then each $U[p_i]$ contains one and only one fixed point of g and as a consequence of Theorem 3.7 each fixed point of g must be essential.

To establish theorems showing the existence of essential fixed points it is usually necessary to put further restrictions on the space X. However for every compact Hausdorff space with the fixed point property the following theorem is true.

Theorem 3.9: If f has a unique fixed point p then p is essential.

Proof: Let W be a neighborhood of p. Choose $U \in B(X)$ such that $U[p] \subset W$. Since F is USC there is a neighborhood N of f such that if g is in N then $F(g) \subset U[F(f)] = U[p] \subset W$. Hence p is essential.

Let E(a,b) denote a compact connected Hausdorff space with exactly two non-cut points a and b. E(a,b) can be simply ordered and the order topology on E(a,b) is the same as the given topology. Since E(a,b) is compact Hausdorff this topology is the uniform topology. Let B(E) denote a base for the uniformity for E(a,b). Assume that $a < b$ in the order on E(a,b). A discussion of this space is found in Hocking and Young (2, p. 52).

The following lemma can be proved using the fact that the Dedekind cut theorem holds in E(a,b).

Lemma 3.3: E(a,b) is locally connected.

Lemma 3.4: Let U be a connected subset of E(a,b). Let f be a continuous mapping on E(a,b) into E(a,b). If there are points l and r in U such that $f(l) \leq l$ and $f(r) \geq r$, then there is a point $p \in U$ such that $f(p) = p$.

<u>Proof</u>: If $f(l) = l$ or $f(r) = r$ then there is nothing to prove. Assume therefore that $f(x) \neq x$ for each $x \in U$. Let $L(f) = \{x \in U | f(x) < x\}$ and $R(f) = \{x \in U | f(x) > x\}$. $L(f)$ and $R(f)$ are non-empty since $l \in L(f)$ and $r \in R(f)$. Furthermore $L(f) \cap R(f) = \emptyset$ and from the continuity of f it follows that $L(f)$ and $R(f)$ are open. The assumption that $f(x) \neq x$ for $x \in U$ implies that $L(f) \cup R(f) = U$. But this contradicts the connectedness of U and hence there is a point $p \in U$ such that $f(p) = p$.

<u>Theorem 3.10</u>: $E(a,b)$ has the fixed point property.

<u>Proof</u>: Since $f(a) \geq a$ and $f(b) \leq b$ the theorem follows from Lemma 3.4.

<u>Theorem 3.11</u>: Let f be a continuous mapping on $E(a,b)$ into $E(a,b)$. Let p be a fixed point of f.

(i) Suppose $p \neq a,b$. If every connected neighborhood of p contains points l and r such that $f(l) < l$ and $f(r) > r$ then p is essential.

(ii) Suppose $p = a$ (respectively, b). If every connected neighborhood U of $a(b)$ contains a point $l(r)$ such that $f(l) < l$ ($f(r) > r$) then p is essential.

<u>Proof of (i)</u>: Let V be a neighborhood of p. By Lemma 3.3 there exists a connected neighborhood V' of p such that

$V' \subset V$. By hypothesis there exist points l and r in V' such that $f(l) < l$ and $f(r) > r$. Choose neighborhoods M of $f(l)$ and N of $f(r)$ such that for every $x \in M$, $x < l$ and for every $x \in N$, $x > r$. Choose U and $U' \in B(E)$ such that $U[f(l)] \subset M$ and $U'[f(r)] \subset N$. Let $U'' = U \cap U'$ and $g \in W(U'')[f]$. Then $g(l) \in U''[f(l)] \subset M$, $g(r) \in U''[f(r)] \subset N$ and it follows that $g(l) < l$ and $g(r) > r$. By Lemma 3.4 g has a fixed point in V and so p is essential. The proof of (ii) is similar.

<u>Theorem 3.12</u>: Let f be a continuous mapping on $E(a,b)$ into itself. If the set of fixed points of f is totally disconnected then at least one fixed point is essential.

<u>Proof</u>: Let $A = \{x|f(x) > x\}$ and $B = \{x|f(x) \leq x\}$. Suppose $A = \emptyset$. Then $f(a) = a$ and every neighborhood of a contains points of B such that $f(x) < x$ since $F(f)$ is totally disconnected. By Theorem 3.11 (ii) a is essential. Suppose $A \neq \emptyset$. Then A has a least upper bound p which must necessarily be a fixed point of f. If $p = b$ then every neighborhood of P contains points of A and again by Theorem 3.11 (ii) p is essential. If $p \neq b$ then every neighborhood of p must contain points of A and points of B which are not fixed. By Theorem 3.11 (i) p is essential.

The "long line" (2, p. 55) is a specific example of a non-metric space to which Theorems 3.10, 3.11 and 3.12 apply.

IV. ESSENTIAL COMPONENTS

Let X be Euclidean and $f \in X^X$. If C is a component of $F(f)$ and contains more than one point then O'Neill (7) shows that no proper subset of C is essential. Since an essential fixed point is an essential singleton set in the sense of O'Neill's definition it is easy to find mappings which have no essential fixed points. A simple example is obtained by taking the identity mapping on a Euclidean n-cell.

Mappings which have no essential fixed points may however have a component of its set of fixed points which has the property of being essential in the following sense:

<u>Definition 4.1</u>: Let $f \in X^X$. Let C be a component of $F(f)$. C is an essential component of $F(f)$ if corresponding to each neighborhood U of C there is a neighborhood N of f such that if $g \in N$ then g has a fixed point in U.

The result obtained in this chapter is that for a certain class of spaces, including non-metric spaces, every self mapping f has at least one essential component C of $F(f)$. This generalizes a result of Kinoshita (5) for separable metric spaces.

<u>Definition 4.2</u>: Let Σ be a non-empty indexing set. For each $\alpha \in \Sigma$, let I_α be the unit interval. The product ΠI_α, $\alpha \in \Sigma$, with the product topology, is called the Tychonoff cube.

Denote this space by T.

For each $x \in T$, x is an α-tuple $x = \{x_\alpha\}$, where $\alpha \in \Sigma$ and $x_\alpha \in I_\alpha$.

Definition 4.3: Let A be a finite subset of Σ and δ a real number such that $0 < \delta \leq 1$. Define

$$U(A,\delta) = \{(x,y) \in T \times T \mid |x_\alpha - y_\alpha| < \delta, \text{ for each } \alpha \in A\}.$$

The family $\{U(A,\delta)\}$ for $A \subset \Sigma$, $\delta \in (0,1]$, is a base for a uniformity for T consisting of open symmetric neighborhoods of the diagonal. Denote this base by B(T). The neighborhood system at $x \in T$ is the family of subsets

$$\begin{aligned} U(A,\delta)[x] &= \{y \mid (x,y) \in U(A,\delta)\} \\ &= \{y \mid |x_\alpha - y_\alpha| < \delta, \text{ for each } \alpha \in A\}. \end{aligned}$$

With this topology T is a compact Hausdorff space.

The following theorem is due to Tychonoff (9).

Theorem 4.1: The Tychonoff cube has the fixed point property.

Definition 4.4: A subset A of a space B is a retract of B provided that there is a continuous mapping r of B into A, such that $r(x) = x$ for each $x \in A$. The mapping r is called a retraction.

Definition 4.5 and Theorem 4.2 can be found in a paper by Saalfrank (8) in which he develops a theory of retracts for normal Hausdorff spaces.

Definition 4.5: A space A is called an absolute retract (AR) provided it is a normal Hausdorff space and for every topological image A_1 of A such that A_1 is a closed subset of a normal Hausdorff space M, it is true that A_1 is a retract of M.

Theorem 4.2: A necessary and sufficient condition for a space to be an AR is that it be homeomorphic to a retract of some Tychonoff cube.

Since T is compact it follows that every AR is compact.

Definition 4.6: Let X be a compact Hausdorff space with the fixed point property. X has property F' if and only if every continuous mapping on X into X has at least one essential component of its set of fixed points.

Theorem 4.3: Property F' is invariant under retraction.

Proof: Let X have property F' and Y be a retract of X under a retraction r. Let f be a continuous mapping on Y into Y. Then fr is a continuous mapping on X into X and since X has property F', F(fr) contains an essential component C. The mapping fr maps X into Y and hence $F(fr) \subset Y$. In fact, $F(fr) = F(f)$ and C is a component of F(f). Let U be a neighborhood of C in Y. Then there is a neighborhood U' of C in X such that $U' \cap Y = U$. Since C is essential there is a neighborhood N of $fr \in X^X$ such that if $g' \in N$ then g' has a fixed point in U'. Choose $V \in B(X)$ such that $W(V)[fr] \subset N$. Let $g \in Y^Y$ such that $g \in W(V')[f]$, where $V' = V \cap Y \times Y$ is a member of the relative uniformity for Y. (See Definition 2.10). Consider now the mapping $gr \in X^X$. Let $x \in X - Y$ and let $r(x) = y \in Y$. Then $gr(x) = g(y) \in V'[f(g)] \subset V[f(y)] = V[fr(x)]$. Similarly if $x \in Y$, then

$$gr(x) = g(x) \in V'[f(x)] \subset V[f(x)] = V[fr(x)].$$

Hence $gr \in W(V)[fr] \subset N$. Consequently gr has a fixed point p in U'. Moreover since gr maps X into Y, $p \in U \subset Y$. Now $gr(p) = p$ and since $r(p) = p$ it follows that $g(p) = p$. Therefore C is an essential component of $F(f)$ and Y has property F'.

<u>Theorem 4.4</u>: Property F' is invariant under a homeomorphism.

<u>Proof</u>: Suppose X has property F'. Let h be a homeomorphism on X to Y and let $f \in Y^Y$. Then $h^{-1}fh$ is a continuous mapping on X into X and since X has property F', $F(h^{-1}fh)$ contains an essential component C. Since $h^{-1}fh(C) = C$ it follows that $f(h(C)) = h(C)$ and that $h(C)$ is a component of $F(f)$ in Y. Let U be a neighborhood of $h(C)$ in Y. Then $h^{-1}(U)$ is a neighborhood of C in X. Since C is essential with respect to $h^{-1}fh$, there is a neighborhood N of $h^{-1}fh \in X^X$, such that if $g \in N$ then g has a fixed point in $h^{-1}(U)$. Since property F' is defined only for compact Hausdorff spaces it follows that Y is also compact Hausdorff and that h is a uniform homeo-

morphism. Let $B(Y)$ denote a base for the uniformity for Y. Choose a $V \in B(X)$ such that $W(V)[h^{-1}fh] \subset N$. Since h^{-1} is uniformly continuous there is a $V' \in B(Y)$ such that $(x,y) \in V'$ implies that $(h^{-1}(x), h^{-1}(y)) \in V$. Choose $g \in Y^Y$ such that $g \in W(V')[f]$ and consider the mapping $h^{-1}gh \in X^X$. For each $x \in X$, $h(x) \in Y$ and $(fh(x), gh(x)) \in V'$. Hence $(h^{-1}fh(x), h^{-1}gh(x)) \in V$ which implies that $h^{-1}gh \in W(V)[h^{-1}fh]$. Therefore $h^{-1}gh$ has a fixed point p in $h^{-1}(U)$. Since $h^{-1}gh(p) = p$ it follows that $gh(p) = h(p)$ where $h(p) \in U$. Therefore g has a fixed point in U and $h(C)$ is an essential component of $F(f)$. Consequently Y has property F'.

Before proving that the Tychonoff cube has property F' it is necessary to have the following theorems and lemmas.

Theorem 4.5: If C is a compact component of a locally compact Hausdorff space S, and U is an open set containing C, then S is the union of disjoint open sets M and N such that $C \subset M \subset U$.

The proof of this theorem can be found in Wilder (10, p. 100).

Lemma 4.1: Let K be a closed subset of a compact Hausdorff space X. Let C be a component of K and U an open set containing C. Then U contains an open set V containing C such that the boundary of V does not intersect K.

Proof: Since K is closed it is a compact Hausdorff subspace of X. Let C be a component of K and $U \subset X$ an open set containing C. Assume that the boundary of U contains points of K. $U' = U \cap K$ is an open set in K containing C. According to Theorem 4.5 there exist sets M and N open in K such that $C \subset M \subset U'$, $M \cap N = \emptyset$ and $K = M \cup N$. In X, M and N are closed and disjoint and hence there are disjoint open sets V and W such that $M \subset V$ and $N \subset W$. Since $C \subset M \subset V$ and $M \subset U' \subset U$, $V \cap U$ is an open set in X containing C whose boundary contains no points of K. For suppose that the boundary of $V \cap U$ contained a point $p \in K$. Then p would have to be a point of N in $\overline{V}$ which is not possible since $N \subset W$ and $W \cap V = \emptyset$.

The next theorem is found in Hocking and Young (2, p. 36).

<u>Theorem 4.6</u>: Let ΠX_α and ΠY_α be two product spaces over the same index set A, and let f_α be a continuous mapping on X_α into Y_α for each $\alpha \in A$. Then the mapping $f(x) = y$, $x = \{x_\alpha\}$, $y = \{f_\alpha(x_\alpha)\}$ is continuous.

<u>Definition 4.7</u>: (i) Let $M = \{(x,y) \in T \times T \mid x_\alpha + y_\alpha \leq 1,$ $\alpha \in \Sigma\}$. Let $(x,y) \in M$. Define

$x + y = \{x_\alpha + y_\alpha\}$, $\alpha \in \Sigma$.

(ii) Let $a \in I = \{x \mid 0 \leq x \leq 1\}$, $x \in T$. Define

$a \cdot x = \{a\, x_\alpha\}$, $\alpha \in \Sigma$.

<u>Lemma 4.2</u>: (i) "+" is a continuous function on M into T. (ii) "$\cdot$" is a continuous function on I x T into T.

The proof of the lemma follows from Theorem 4.6.

Kelley proves the following theorem (4, p. 91).

<u>Theorem 4.7</u>: A function f on a topological space to a product space $\Pi\{X_\alpha | \alpha \in A\}$ is continuous if and only if the composition $P_\alpha \cdot f$ is continuous for each projection P_α.

<u>Lemma 4.3</u>: Let f and g be in T^T and a,b $\in I^T$. If for each $x \in T$, $a(x)\cdot f(x) + b(x)\cdot g(x) \in T$, then the function h defined by $h(x) = a(x)\cdot f(x) + b(x)\cdot g(x)$ is in T^T.

<u>Proof</u>: Let G be a function on T into $I^2 \times T^2$ defined by $G(x) = (a(x), b(x), f(x), g(x))$. Let F be a function on $I^2 \times T^2$ into T defined by $F(p,q,y,z) = p\cdot y + q\cdot z$. F is continuous since "+" and "$\cdot$" are continuous by Lemma 4.2. Since the compositions $P_1\cdot G = a$, $P_2\cdot G = b$, $P_3\cdot G = f$ and $P_4\cdot G = g$ are continuous it follows from Theorem 4.7 that G is continuous. Now h can be written as the composition of the functions F and G, and consequently h is continuous.

<u>Theorem 4.8</u>: The Tychonoff cube has property F'.

<u>Proof</u>: Let $f \in T^T$. Let F(f) be decomposed into components C_α, $\alpha \in \Gamma$, such that $F(f) = \bigcup C_\alpha$, $\alpha \in \Gamma$. Each C_α is compact and $C_\alpha \cap C_\beta = \emptyset$ if $\alpha \neq \beta$. Assume that no C_α is essential. Then for each C_α there is a neighborhood U_α

such that every neighborhood of f contains a mapping g_α which has no fixed point in U_α. According to Lemma 4.1 there exists a neighborhood U'_α for each C_α such that $U'_\alpha \subset U_\alpha$ and such that the boundary of U'_α does not intersect $F(f)$. Since $F(f)$ is compact it is possible to select a finite set $\{U'_{\alpha_i}\}$, $i = 1, 2, \ldots, n$, which covers $F(f)$. It follows that each $C_\alpha \subset U'_{\alpha_i} \subset U_{\alpha_i}$ for some i, $1 \leq i \leq n$, and in particular $C_{\alpha_i} \subset U'_{\alpha_i} \subset U_{\alpha_i}$. Now define

$$F_1 = F(f) \cap \overline{U'_{\alpha_1}} \quad \text{and}$$

$$F_i = F(f) \cap [\overline{U'_{\alpha_i}} - \bigcup_{j<i} U_{\alpha_j}];\ i = 2, 3, \ldots, n.$$

Each F_i is closed and $F_i \cap F_j = \emptyset$ for $i \neq j$. Hence there exist open sets V_i and W_i, $i = 1, 2, \ldots, n$, such that

$$F_i \subset W_i \subset \overline{W_i} \subset V_i \subset U'_{\alpha_i} \text{ and such that}$$

$$V_i \cap V_j = \emptyset \text{ for } i \neq j.$$

Since f has no fixed points in the compact subset $T - \bigcup_{i=1}^{n} W_i$, according to Lemma 3.1, there is a $U(A,\delta) \in B(T)$ such that $(x,f(x)) \notin U(A,\delta)$ for each $x \in T - \bigcup_{i=1}^{n} W_i$. Since $C_{\alpha_i} \subset F_i \subset V_i \subset U_{\alpha_i}$, it is possible to find a $g_i \in W(U)[f]$, where $U = U(A,\delta)$, such that g_i has no fixed point in V_i. Since $\bigcup_{i=1}^{n} \overline{W_i}$ and $T - \bigcup_{i=1}^{n} V_i$ are closed disjoint subsets of a normal space there is a continuous function a on T to I such that $a(x) = 0$ on $\bigcup_{i=1}^{n} \overline{W_i}$ and $a(x) = 1$ on $T - \bigcup_{i=1}^{n} V_i$. Let $b(x) = 1 - a(x)$. A mapping $g \in T^T$ will now be constructed which has no fixed point in T.

Define

$$g(x) = f(x) \text{ for } x \in T - \bigcup_{i=1}^{n} V_i \quad ;$$

$$= g_i(x) \text{ for } x \in \overline{W_i} \quad ; \text{ and}$$

$$= a(x)\cdot f(x) + b(x)\cdot g_i(x) \text{ for } x \in \overline{V_i} - W_i \quad .$$

For convenience let $f(x) = \{x'_\alpha\}$, $g_i(x) = \{x^i{}_\alpha\}$ and $g(x) = \{x''_\alpha\}$, $\alpha \in \Sigma$. With this notation the last equation can be written as

$$g(x) = \{x''_\alpha\} = \{a(x)\ x'_\alpha + b(x)\ x^i{}_\alpha\}.$$

By construction and in view of Lemma 4.3 g is continuous on T into T. It remains to be shown that g has no fixed point in T.

f has no fixed point in $T - \bigcup_{i=1}^{n} V_i$ and g_i has no fixed point in $\overline{W_i}$, $i = 1, 2, \ldots, n$. Hence from the definition of g, g has no fixed points in those regions. For each $x \in \bigcup_{i=1}^{n} [\overline{V_i} - W_i]$, it will now be shown that

$(g(x), f(x) \in U(A,\delta)$ and since $(x, f(x)) \notin U(A,\delta)$, it will follow that $g(x) \neq x$ in this region and hence in T. Let $x \in \overline{V_i} - W_i$. For each $\alpha \in A$,

$$\begin{aligned} |x''_\alpha - x'_\alpha| &= |a(x)\, x'_\alpha + b(x)\, x^i_\alpha - x'_\alpha| \\ &= b(x)|x'_\alpha - x^i_\alpha| \\ &\leq |x'_\alpha - x^i_\alpha| < \delta. \end{aligned}$$

The last inequality follows from the fact that $(g_i(x), f(x)) \in U(A,\delta)$, $i = 1, 2, \ldots, n$. Consequently $(g(x), f(x)) \in U(A,\delta)$ for $x \in \bigcup_{i=1}^{n} [\overline{V_i} - W_i]$ and in view of the above discussion g has no fixed point in T. This contradicts the fact that T has the fixed point property and therefore the assumption that no C_α is essential if false.

<u>Theorem 4.9</u>: Every AR has property F'.

<u>Proof</u>: Let A be an AR. Then by Theorem 4.2 A is homeomorphic to a retract A_1 of T. From Theorems 4.7, 4.3 and 4.4 it follows respectively that T, A_1 and hence A have property F'.

Let X be a compact Hausdorff space with the fixed point property. Then the following theorem holds.

<u>Theorem 4.10</u>: Let $f \in X^X$. If F(f) is connected then F(f) = C is an essential component.

Proof: Let U be a neighborhood of C. By Theorem 2.8 there is a $V \in B(X)$ such that $V[C] \subset U$. Since F is USC there is a neighborhood N of f such that if $g \in N$ then $F(g) \subset V[F(f)] = V[C]$. Hence C is essential.

If $C = \{p\}$ then this theorem reduces to Theorem 3.9.

V. LOCAL STUDY OF ESSENTIAL COMPONENTS

Various theorems in the preceding chapters have been established giving sufficient conditions for existence of essential fixed points or components. However when a mapping has more than one component in its set of fixed points these theorems are of no use in determining those components which are essential. It is therefore desirable to have theorems which may be used to prove essentiality by examining the behavior of the mapping in a neighborhood of the component. One such theorem follows.

<u>Theorem 5.1</u>: Let X be a compact Hausdorff space. Let $f \in X^X$ and C be a component of $F(f)$. If every neighborhood U of C contains an open neighborhood V of C such that $\overline{V} \subset U$, $\overline{V}$ has the fixed point property and $f(\overline{V}) \subset V$, then C is essential.

<u>Proof</u>: Let U be a neighborhood of C. Choose an open neighborhood V of C such that $\overline{V} \subset U$ and such that $\overline{V}$ has the fixed point property. Since $f(\overline{V}) \subset V$ there is a $U' \in B(X)$ such that $U'[f(\overline{V})] \subset V$. Choose $g \in X^X$ such that $g \in W(U')[f]$. Then $g(x) \in U'[f(x)]$ for every $x \in X$ and hence $g(x) \in U'[f(\overline{V})]$ for $x \in \overline{V}$. Consequently $g(\overline{V}) \subset U'[f(\overline{V})] \subset V$ and since $\overline{V}$ has the fixed point property, g has a fixed point in $\overline{V}$ and hence in U. Therefore C is essential.

Note that in Theorem 5.1 X is not required to have the fixed point property and it is necessary for g to be "close" to f only on $\overline{V}$ and not on the whole space. For many spaces the essentiality of a component of the set of fixed points depends only on the nature of the mapping in a neighborhood of the component. O'Neill (7) proves that this is the case if X is a polyhedral space. Hence it is true if X is Euclidean. The next theorem shows that the Tychonoff cube has this local property.

<u>Theorem 5.2</u>: Let C be a component of F(f), $f \in T^T$. Let U be an open neighborhood of C and let $g \in T^T$ agree with f on U. Then C is an essential component of F(g) if and only if C is an essential component of F(f).

<u>Proof</u>: Let U be an open neighborhood of C and let f = g on U. C is a component of F(g). For suppose $C \subset C'$ where C' is a component of F(g). Then U contains fixed points of g which are not fixed points of f and contradicts the fact that f = g on U.

Assume that C is essential with respect to f but inessential with respect to g. Then there is an open set U' containing C such that every neighborhood of g contains a mapping which has no fixed point in U'. Let V be an open neighborhood of C such that $\overline{V} \subset U \cap U'$. Since C is essential with respect to f there is a neighborhood N of f such that if

$f' \in N$ then f' has a fixed point in V. Choose $V' = V'(A,\delta) \in B(T)$ such that $W(V')[f] \subset N$. Choose $h \in T^T$ such that $h \in W(V')[g]$ and such that h has no fixed point in U'. Let $a \in I^T$ such that $a(x) = 0$ for $x \in \overline{V}$ and $a(x) = 1$ for $x \in T - U \cap U'$. Let $b(x) = 1 - a(x)$. Construct a mapping h' as follows: Let

$$\begin{aligned} h'(x) &= f(x) \text{ for } x \in T - U \cap U' \quad ; \\ &= h(x) \text{ for } x \in \overline{V} \quad ; \text{ and} \\ &= a(x)\cdot f(x) + b(x)\cdot h(x) \text{ for } x \in \overline{U \cap U'} - V \quad . \end{aligned}$$

By construction and Lemma 4.3, $h' \in T^T$. Furthermore, as in the proof of Theorem 4.7, it can be shown that $h' \in W(V')[f]$. Hence h' must have a fixed point in V. But $h' = h$ on $\overline{V}$ and h was chosen with no fixed point in U'. Since $V \subset U'$, h' has no fixed point in V which contradicts the essentiality of C with respect to f. Thus C is an essential component of $F(g)$.

A similar argument shows that if C is essential with respect to g then it is also essential with respect to f.

In conclusion an example is given of a mapping $f \in T^T$ which has two fixed points. It is shown that one is essential and the other inessential. The following lemmas are used in the proof.

<u>Lemma 5.1</u>: Let h be a mapping of ΠX_α onto ΠY_α, $\alpha \in \Sigma$,

defined as follows: For each $\alpha \in \Sigma$, let h_α be a homeomorphism of X_α onto Y_α. Then h is a homeomorphism.

<u>Proof</u>: h is continuous by Theorem 4.6. Let x and y be in T and suppose that $h(x) = h(y)$. Then $h_\alpha(x_\alpha) = h_\alpha(y_\alpha)$ for each $\alpha \in \Sigma$ and since h_α is one to one it follows that $x_\alpha = y_\alpha$ and that $x = y$. Hence h is one to one. h^{-1} is defined by $h^{-1}{}_\alpha$ on Y_α to X_α. Since $h^{-1}{}_\alpha$ is continuous for each α, again by Theorem 4.6, h^{-1} is continuous. Therefore h is a homeomorphism.

<u>Lemma 5.2</u>: Let $x \in T$ and V be a neighborhood of x. Then there is a $U = U(A,\delta) \in B(T)$ such that $\overline{U[x]} \subset V$ and $\overline{U[x]}$ has the fixed point property.

<u>Proof</u>: Choose $U = U(A,\delta) \in B(T)$ such that $\overline{U[x]} \subset V$. Let $I'_\alpha = \{y_\alpha \in I_\alpha \mid |y_\alpha - x_\alpha| \leq \delta,\ \alpha \in A\}$. Then

$$\overline{U[x]} = \{y \mid |y_\alpha - x_\alpha| \leq \delta,\ \alpha \in A\} = \prod_{\alpha\in A} I'_\alpha \times \prod_{\alpha\in\Sigma-A} I_\alpha .$$

Define a mapping h on T onto $\overline{U[x]}$ as follows: For each $\alpha \in A$, let h_α be a homeomorphism of I_α onto I'_α. If $\alpha \in \Sigma - A$, let h_α be the identity on I_α. Then

$h(T) = \prod_{\alpha\in A} I'_\alpha \times \prod_{\alpha\in\Sigma-A} I_\alpha = \overline{U[x]}$ and by Lemma 5.1, h is a homeomorphism. Since the fixed point property is a topological property it follows that $\overline{U[x]}$ has the fixed point property.

Example: Let f be a mapping on T into T defined as follows. For each $\alpha \in \Sigma$, let f_α be defined by $f_\alpha(x_\alpha) = x^2_\alpha$. By Theorem 4.6 f is continuous. Let 0 denote the point $x = \{x_\alpha\} \in T$ such that $x_\alpha = 0$ for each $\alpha \in \Sigma$. Let 1 denote the point $x \in T$ such that $x_\alpha = 1$ for each $\alpha \in \Sigma$. Then it is clear that $f(0) = 0$ and $f(1) = 1$. According to Theorem 4.7 at least one of these fixed points must be essential.

Let V be a neighborhood of $x = 0$. Then by Lemma 5.2 there is a $U = U(A,\delta) \in B(T)$ such that $\overline{U[0]} \subset V$ and $\overline{U[0]}$ has the fixed point property. Assume that $\delta < 1$. Let $y \in \overline{U[0]}$. Then $y_\alpha \leq \delta < 1$, $\alpha \in A$. If $y_\alpha \neq 0$, then $f_\alpha(y_\alpha) = y^2_\alpha < y_\alpha \leq \delta$. If $y_\alpha = 0$, then $f_\alpha(0) = 0 < \delta$. Hence for each $\alpha \in A$, $f_\alpha(y_\alpha) < \delta$ and therefore $f(y) \in U[0]$ and so $f(\overline{U[0]}) \subset U[0]$. From Theorem 5.1 it follows that $x = 0$ is an essential fixed point of f.

Choose $U = U(A, 1/2) \in B(T)$ and let $U[1]$ be the corresponding neighborhood of $x = 1$. For each λ, $0 < \lambda \leq 1$, define a continuous mapping on T into T as follows. For each $\alpha \in \Sigma$, let $g_\alpha(x_\alpha) = (1 - \lambda)x^2_\alpha$. The only fixed point of each g_α is $x_\alpha = 0$ and hence it is true that the only fixed point of g_λ is $x = 0$. Let N be any neighborhood of f. Then there is a $U' = U'[A', \delta'] \in B(T)$ such that $W(U')[f] \subset N$. Choose g_λ such that $\lambda < \delta'$. Then for each $\alpha \in A'$,

$$|g_\alpha(x_\alpha) - f_\alpha(x_\alpha)| = |(1 - \lambda)x^2_\alpha - x^2_\alpha| = \lambda x^2_\alpha \leq \lambda < \delta' .$$

Hence $g_\lambda \epsilon W(U')[f]$ but g_λ has no fixed point in $U[1]$ since $x = 0 \notin U[1]$. Therefore for every neighborhood N of f there is a $g_\lambda \epsilon N$ such that g_λ has no fixed point in $U[1]$. Hence $x = 1$ is an inessential fixed point of f.

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VII. ACKNOWLEDGEMENT

The author wishes to express his appreciation to Dr. D. E. Sanderson for his encouragement, patience and guidance during the preparation of this thesis.